

Step 3. $\forall \sigma \neq \text{id} : \sigma(a) \sigma(b) < 0$

Know: $\Delta(H_0)$ invad lattice in $\prod_{\sigma \in S^\infty} H^\sigma$ and proj. in H^{id} lattice

$\Rightarrow H^\sigma$ cpt $\forall \sigma \neq \text{id} \rightarrow H^0 \cong \text{SO}(2,1) \quad \forall \sigma \neq \text{id}$

$\Rightarrow \sigma(a), \sigma(b), \sigma(-1)$ have the same sign $\Rightarrow \sigma(a), \sigma(b) < 0$.

Prop. $\text{SL}(2, \mathbb{Z})$ is the only non-cppt arith subgp of $\text{SL}(2, \mathbb{R})$ up to conjugation and commensurability.

Pf. Work with $\text{SO}(2,1)$. Goal: prove that $\text{SO}(2,1)_{\mathbb{Z}}$ is the "only" non-cppt arith subgp of $\text{SO}(2,1)$.

Know: $\exists G < \text{SL}(l, \mathbb{R})$ simple for some l s.t. G is defined over some number field F and $\exists \varphi : G \rightarrow \text{SO}(2,1)$ isogeny s.t. $\varphi(G_{O_F})$ is comm to Γ .

Pf of prev. Prop. $\Rightarrow G = \text{SO}(B, \mathbb{R})$ where $B : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ bilinear form of signature $(2,1)$.

NTS: G is defined over \mathbb{Q} .

Recall from the original def of arith gps: $\exists K'$ cpt s.t.

$$G' = G \times K' < \text{SL}(l, \mathbb{R}), \quad G' \text{ defined over } \mathbb{Q}, \quad \Gamma K' = G'_{\mathbb{Z}} K'$$

$N =$ Zariski closure of unipotent elts in $G'_{\mathbb{Z}}$

K' cpt. $\Rightarrow N \subseteq G$

N is normalised by $G'_{\mathbb{Z}} \xrightarrow[\text{density}]{\text{Borel}} N \triangleleft G \xrightarrow{G \text{ simple}} N = G$.

G is the Zariski closure of a group defined over $\mathbb{Z} \Rightarrow G$ is def. / \mathbb{Q} .

Step 2. Have: $G = \text{SO}(B, \mathbb{R})$. Want: $B(x,x) = x_1^2 + x_2^2 - x_3^2$

Γ not cppt $\Rightarrow B$ is isotropic over $F \Rightarrow \exists u \in F^3 \setminus \{0\} : B(u,u) = 0$.

Choose $v \in F^3 \setminus \{0\}$ s.t. $B(u,v) \neq 0$

By adding a multiple of u wma $B(v,v) = 0$.

Choose $w \in F^3 \setminus \{0\}$ s.t. $B(w,u) = B(w,v) = 0$.

Multiplying B with scalars preserves everything \Rightarrow wma $B(w,w) = 2B(v,v) = 1$.

$\rightarrow B$ has the desired form wrt. the basis $\{w, u+v, u-v\}$.

This week we again describe the arithmetic subgroups of $SL(2, \mathbb{R})$ via quaternion algebras.

Def. Let F be a field, $a, b \in F^\times$. Then the corresponding quaternion algebra is the ring
$$\mathbb{H}_F^{a,b} = \left\{ p + qi + rj + sk \mid p, q, r, s \in F \right\}$$

with addition coming from F and multiplication determined by $i^2 = a, j^2 = b, ij = -ji = k$ with F lying in the center of $\mathbb{H}_F^{a,b}$.

Def. The reduced norm $N_{red}: \mathbb{H}_F^{a,b} \rightarrow F$ is defined as

$$N_{red}(p + qi + rj + sk) = (p + qi + rj + sk) \cdot \overline{(p + qi + rj + sk)} = p^2 - aq^2 - br^2 + abs$$

where the conjugation is defined as $\overline{p + qi + rj + sk} = p - qi - rj - sk$.

Note that $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$ and $\mathbb{H}_F^{a,b} \cong \mathbb{H}_F^{b,a}$ (flipping i and j).

Ex. 1) $\mathbb{H}_{\mathbb{R}}^{-1,-1} = \mathbb{H}$ Hamiltonian quaternions

2) $\mathbb{H}_F^{t^2a, t^2b} \cong \mathbb{H}_F^{a,b} \quad \forall a, b, t \in F^\times$ (Exc.)

3) $\mathbb{H}_F^{a^2, b} \cong \text{Mat}_{2 \times 2}(F) \quad \forall a, b \in F^\times$

The isomorphism $\varphi: \mathbb{H}_F^{a,b} \rightarrow \text{Mat}_{2 \times 2}(F)$ is given by

$$\varphi(1) = \text{id}, \quad \varphi(i) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad \varphi(j) = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}, \quad \varphi(k) = \begin{pmatrix} 0 & a \\ -ba & 0 \end{pmatrix}.$$

Lemma. $\mathbb{H}_{\mathbb{C}}^{a,b} \cong \text{Mat}_{2 \times 2}(\mathbb{C}) \quad \forall a, b \in \mathbb{C}^\times$ and $\mathbb{H}_{\mathbb{R}}^{a,-1} \cong \begin{cases} \text{Mat}_{2 \times 2}(\mathbb{R}) & a > 0 \\ \mathbb{H} & a < 0 \end{cases}$

PF: Follows from 3). Exc: show $\text{Mat}_{2 \times 2}(\mathbb{R}) \not\cong \mathbb{H}$. □

Prop. $a, b > 0$ integers. $SL(1, \mathbb{H}_{\mathbb{R}}^{a,b}) := \left\{ g \in \mathbb{H}_{\mathbb{R}}^{a,b} \mid N_{red}(g) = 1 \right\}$

Then 1) $SL(1, \mathbb{H}_{\mathbb{R}}^{a,b}) \cong SL(2, \mathbb{R})$

2) $SL(1, \mathbb{H}_{\mathbb{R}}^{a,b})_{\mathbb{Z}} < SL(1, \mathbb{H}_{\mathbb{R}}^{a,b})$ is an arithmetic subgrp.

where $SL(1, \mathbb{H}_{\mathbb{R}}^{a,b})_{\mathbb{Z}} = \left\{ x_0 + x_1i + x_2j + x_3k \in SL(1, \mathbb{H}_{\mathbb{R}}^{a,b}) \mid \forall x_i \in \mathbb{Z} \right\}$.

3) TFAE:

(a) $SL(1, \mathbb{H}_{\mathbb{R}}^{a,b})_{\mathbb{Z}} < SL(1, \mathbb{H}_{\mathbb{R}}^{a,b})$ is cocompact

(b) $(0, 0, 0, 0)$ is the only integral solution of $w^2 - ax^2 - by^2 + abz^2 = 0$

(c) Every $\neq 0$ elt in $\mathbb{H}_{\mathbb{Q}}^{a,b}$ has a multiplicative inverse, i.e.

$\mathbb{H}_{\mathbb{Q}}^{a,b}$ is a division algebra.

PF: 1) Know: $\mathbb{H}_{\mathbb{R}}^{a,b} \cong \text{Mat}_{2 \times 2}(\mathbb{R})$ via $\varphi(1) = \text{id}$, $\varphi(i) = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}$,

$$\varphi(j) = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}, \quad \varphi(k) = \begin{pmatrix} 0 & \sqrt{a} \\ -b\sqrt{a} & 0 \end{pmatrix}.$$

NTS: $N_{\text{red}} = 1 \Leftrightarrow \det = 1$, i.e. $N_{\text{red}}(g) = \det(\varphi(g))$.

$$g := p + qi + rj + sk$$

$$\det \varphi(g) = \det \begin{pmatrix} p + q\sqrt{a} & r + s\sqrt{a} \\ br - bs\sqrt{a} & p - q\sqrt{a} \end{pmatrix} = p^2 - aq^2 - b(r^2 - as^2) = N_{\text{red}}(g). \quad \checkmark$$

2) We could use the Prop. from last week and just go through the list.

Or directly: $SL(1, \mathbb{H}_{\mathbb{R}}^{a,b}) \curvearrowright \mathbb{H}_{\mathbb{R}}^{a,b}$,

$\mathbb{H}_{\mathbb{Q}}^{a,b}$ is a \mathbb{Q} -form of $\mathbb{H}_{\mathbb{R}}^{a,b}$

$\mathbb{H}_{\mathbb{Z}}^{a,b}$ is a \mathbb{Z} -lattice of the \mathbb{Q} -form $\mathbb{H}_{\mathbb{Q}}^{a,b}$

and $\gamma(\mathbb{H}_{\mathbb{Z}}^{a,b}) \subseteq \mathbb{H}_{\mathbb{Z}}^{a,b}$ iff $\gamma \in \mathbb{H}_{\mathbb{Z}}^{a,b}$

$\Rightarrow SL(\mathbb{H}_{\mathbb{R}}^{a,b}) \cap \mathbb{H}_{\mathbb{Z}}^{a,b} = SL(\mathbb{H}_{\mathbb{R}}^{a,b})_{\mathbb{Z}}$ is an arithmetic subgroup. \checkmark

3) (c) \Rightarrow (a): We will show that $G_{\mathbb{Z}}$ non-copt $\Rightarrow \mathbb{H}_{\mathbb{Q}}^{a,b}$ not a div. alg.

Here $G = SL(1, \mathbb{H}_{\mathbb{R}}^{a,b})$.

Godement $\Rightarrow G_{\mathbb{Z}}$ has a nontrivial unipotent element

$\exists \gamma \in G_{\mathbb{Z}} \setminus \{1\}$ that has 1 as an eigenvalue, i.e.

$\exists v \in \mathbb{H}_{\mathbb{Q}}^{a,b} \setminus \{0\}$ s.t. $\gamma v = v \Rightarrow (\gamma - 1)v = 0 \Rightarrow (\gamma - 1) \in \mathbb{H}_{\mathbb{Q}}^{a,b} \setminus \{0\}$ is

a zero divisor. \checkmark

(a) \Rightarrow (c): We will prove: if $\mathbb{H}_{\mathbb{Q}}^{a,b}$ is not a div. alg. $\Rightarrow G_{\mathbb{Z}}$ not copt.

Wedderburn's Theorem (PT): if $\mathbb{H}_{\mathbb{Q}}^{a,b}$ is not a div. alg. $\Rightarrow \mathbb{H}_{\mathbb{Q}}^{a,b} \cong \text{Mat}_{2 \times 2}(\mathbb{Q})$.

(Note that WT is more general, this is just a special case.)

For a direct pf, cf. Exc. 6.2.3.)

$\Rightarrow SL(1, \mathbb{H}_{\mathbb{Z}}^{a,b}) \cong SL(2, \mathbb{Z})$, which is not copt.

(b) \Rightarrow (c) Exc. 6.2.4.

(a) \Rightarrow (b) already shown (where?)

The following can be proved in a similar fashion:

Prop. • $F \neq \mathbb{Q}$ totally real number field

• O_F ring of integers

• $a, b \in F^+$, $\sigma(a), \sigma(b) < 0 \quad \forall \sigma \neq \text{id}$

• $G = SL(1, \mathbb{H}_{\mathbb{R}}^{a|b}) \simeq SL(2, \mathbb{R})$

$\Rightarrow G_{O_F} = SL(1, \mathbb{H}_{O_F}^{a|b})$ cocpt. arith. subgp. of G .

Prop. Every cocpt. arith. subgp. of $SL(2, \mathbb{R})$ is of one of the two forms above, up to conjugation and commensurability.

Pf. Recall: the two types are

(1) $a, b > 0$ integers, $(0, 0, 0)$ is the sole integral solution to $ax^2 + by^2 - z^2 = 0$. Then for $G = SO(ax^2 + by^2 - z^2; \mathbb{R})$ we have that $G_{\mathbb{Z}} < G$ is cocpt.

(2) $F \neq \mathbb{Q}$ totally real number field, $a, b \in F^+$, $\sigma(a), \sigma(b) < 0 \quad \forall \sigma \neq \text{id}$. Then $G_{O_F} < G$ is a cocpt. arith. subgp.

We will show that (1) is of the form $SL(1, \mathbb{H}_{\mathbb{Z}}^{a|b})$.

Claim 1 If $\exists g \in \mathbb{H}_{\mathbb{F}}^{a|b} \setminus \{0\}$ s.t. $N_{\text{red}}(g) = 0$ then $\exists g' \in \mathbb{H}_{\mathbb{F}}^{a|b} \setminus \{0\}$ that has k -component 0 and $N_{\text{red}}(g') = 0$

Pf. $g = x_0 + ix_1 + jx_2 + kx_3$, $x_3 \neq 0$, $\alpha := \frac{-x_2}{x_3} + i$
 $\alpha g \neq 0$ but its k -component is 0.

$$N_{\text{red}}(\alpha g) = 0$$

$\Rightarrow \exists (w, x, y, z) \in \mathbb{Q}^4 \setminus \{0\}$ with $N_{\text{red}}(w + xc + yj + zk) = 0$

iff $\exists (x, y, z) \in \mathbb{Q}^3 \setminus \{0\}$ with $ax^2 + by^2 = z^2$.

So if Γ is of the form (1) $\Rightarrow G_{\mathbb{Z}} = SL(1, \mathbb{H}_{\mathbb{Z}}^{a|b}) < SL(2, \mathbb{R}) = G$ is a cocpt. arith. subgp.

Goal: show that Γ and $G_{\mathbb{Z}}$ are commensurable.

\mathfrak{g} = Lie alg of G . This can be viewed as a subspace of $\mathbb{H}_{\mathbb{R}}^{a|b}$:

$$\mathfrak{g} = \left\{ v \in \mathbb{H}_{\mathbb{R}}^{a|b} \mid \text{Re}(v) = 0 \right\}. \quad \text{Exc. 6.2.7.}$$

And $\text{Ad}_G(g)(v) = gv g^{-1}$.

$N_{\text{red}}|_{\mathfrak{g}}$ is a quadratic form on \mathfrak{g} that is invariant under Ad_G .

$$N_{\text{red}}(v) = N_{\text{red}}(xi + yj + zk) = -ax^2 - by^2 + abz^2 \quad \text{for } v = xi + yj + zk \in \mathfrak{g}$$

Change v'bles: $x \mapsto by, y \mapsto ax$:

$$N_{\text{red}}(v) = \underbrace{-ab}(ax^2 + by^2 - z^2)$$

multiplication by a scalar does not change the group.

\Rightarrow arithmetic group of type (1) is commensurable to G_2

Identification of the other two types goes similarly.

14.06.2018

Hyperbolic surfaces

Sources: Beardon The geometry of discrete gps

Buser The gty and spectra of opt Riemann surfaces

Katok Fuchsian gp

Recall: $\mathbb{H}^2 = \text{SO}(2,1)^\circ / \text{SO}(2) = \{z \in \mathbb{C} \mid \text{Im} z > 0\}$ $ds^2 = \frac{dx^2 + dy^2}{y^2}$

Def. Hyperbolic surface: locally symmetric space with universal cover \mathbb{H}^2 .

Closed hyperbolic surface: compact + hyperbolic surface (and has no boundary)



Thm. (19th cty) Classification of surfaces.

Let S be a closed orientable surface. Then S is diffeomorphic to a genus g surface (i.e. # of g tori with S^2) for some $g \geq 0$.

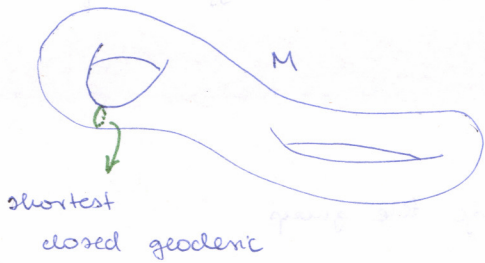
Thm. (Gauss-Bonnet) S can be given the structure of a hyperbolic surface iff it has genus ≥ 2 .

Fact. For fixed genus $g \geq 2$ there is a $6g-6$ dimensional deformation space of hyperbolic structures on a genus g surface.

(Equivalently, there are a lot of lattices in $\text{Isom}^+(\mathbb{H}^2)$ that are isometric to $\pi_1(S_g)$.)

Def. The systole of a Riemannian manifold M is

$$\text{sys}(M) := \inf \{ \text{lengths of closed geodesics in } M \}$$



Fact. On a hyperbolic surface the systole is realised by a closed geodesic.

Question. What is $\max \{ \text{sys}(M) \mid M \text{ hyp. surface of genus } g \}$?

Fact. This is OPEN.

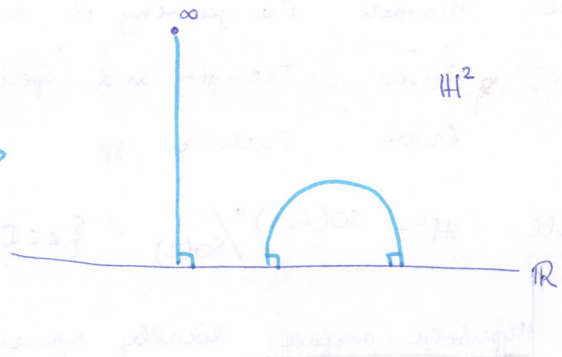
Closed geodesics on hyperbolic surfaces

$$\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \} \quad ds^2 = \frac{dx^2 + dy^2}{y^2}$$

What are the geodesics in \mathbb{H}^2 ?

Follows from the explicit formula for the metric:

- vertical lines and
- semicircles $\perp \mathbb{R}$



Indeed, any $x, y \in \mathbb{H}^2 \nexists!$ geodesic s.t. $x, y \in \gamma$.

Geodesics are parametrised by pairs of points in $\mathbb{R} \cup \{\infty\} = \partial_\infty \mathbb{H}^2$, i.e. their end points

Isometries: $SL(2, \mathbb{R}) \curvearrowright \mathbb{H}^2$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$. This extends to $\partial_\infty \mathbb{H}^2$.

Note: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} z = z \Rightarrow$ get an action of $PSL(2, \mathbb{R}) = SL(2, \mathbb{R}) / \pm Id = \text{Isom}^+(\mathbb{H}^2)$

Classification of elements of $PSL(2, \mathbb{R})$

Def. An isometry is called

- hyperbolic if it has two fixed pts on $\partial_\infty \mathbb{H}^2$
- parabolic if it has one fixed pt on $\partial_\infty \mathbb{H}^2$
- elliptic if it has a fixed pt in \mathbb{H}^2 .

Examples • Hyperbolic: $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in \text{PSL}(2, \mathbb{R})$ for $\lambda > 0$, $z \mapsto \lambda^2 z$

Fixed pts: $0, \infty$

• Parabolic: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $z \mapsto z+1$, fixed pt is ∞ only

• Elliptic: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $z \mapsto \frac{-1}{z}$, has fixed pt i

Fact • All isometries of \mathbb{H}^2 are either hyperbolic, parabolic or elliptic.

• If $g \in \text{PSL}(2, \mathbb{R})$ is hyperbolic then $\exists h \in \text{PSL}(2, \mathbb{R})$ s.t. $hgh^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ for some $\lambda > 0$

• If $g \in \text{PSL}(2, \mathbb{R})$ is parabolic then $\exists h \in \text{PSL}(2, \mathbb{R})$ s.t. $hgh^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

• For elliptic isometries, the situation is slightly more complicated and won't be discussed as we won't need them.

Translation distance

Def: Let $g \in \text{PSL}(2, \mathbb{R})$ be hyperbolic. Then the translation distance of g is

$$T_g := \inf \{ d_{\mathbb{H}^2}(gx, x) \mid x \in \mathbb{H}^2 \}$$

The axis of g is $\alpha_g := \{ x \in \mathbb{H}^2 \mid d_{\mathbb{H}^2}(gx, x) = T_g \}$.

Fact • α_g is the geodesic determined by the 2 fixed pts of g

$\Rightarrow g = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ has axis $\alpha_g = \mathbb{R}_{>0} i$

• $T_g = 2 \cosh^{-1} \left(\frac{|\text{tr}(g)|}{2} \right)$ Note that $\text{tr}(g)$ is determined up to sign only, but $|\text{tr}(g)|$ is well-defined.

Geodesics on hyperbolic surfaces

Prop: S closed hyperbolic surface \Rightarrow in every free homotopy class of closed curves on S there is a unique geodesic minimising length over the ltpy class.

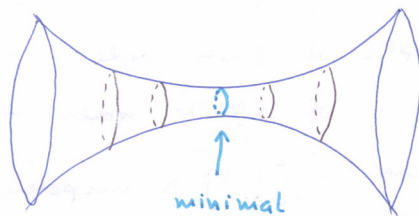
PF (SKETCH): To find a geodesic, use Arzela-

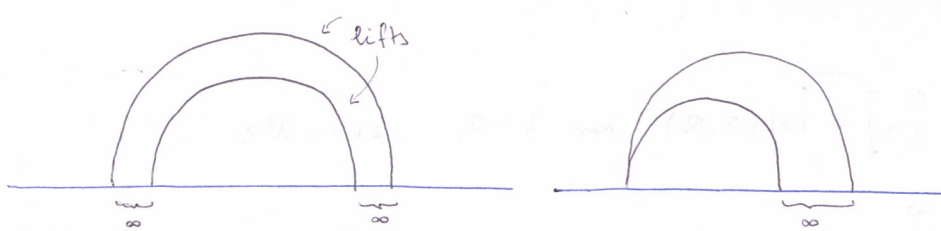
Ascoli argument.

Uniqueness: suppose $\exists 2$ such geodesics,

lift to \mathbb{H}^2 . Since the original geodesics are

of bounded distance from each other, so are their lifts.





If the end pts do not coincide, the distance will not be bounded

→ end pts must be the same. But end pts determine the geodesic

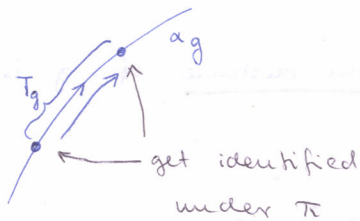
→ the lifts must be the same → project to the same geodesic on S . □

Prop. There is a one-to-one correspondence between

$$\{\text{oriented closed geodesics on } S\} \longleftrightarrow \{\text{conjugacy classes of hyp. elts in } \Gamma\}$$

where $S = \mathbb{H}^2 / \Gamma$. Moreover if g is in the conjugacy class in Γ corresponding to the geodesic γ then $l(\gamma) = Tg$.

Pf (sketch): $g \in \Gamma$ hyperbolic elt. Then α_g projects to a closed geodesic under the covering map $\pi: \mathbb{H}^2 \rightarrow \mathbb{H}^2 / \Gamma$



→ the length of the geodesic is Tg . □

The direction in which g translates gives the orientation

E.g. $g = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ translates upwards along α_g if $\lambda > 1$ and downwards if $\lambda < 1$.

Exc.: $\alpha_{hgh^{-1}} = h(\alpha_g)$

→ they project to the same geodesic. ✓

The other direction is based on a lifting argument. and $\alpha_{hgh^{-1}} = h(\alpha_g)$. □

Conclusion: to control systole, we need to control the lowest trace of hyperbolic elts in Γ .

Note that if S is closed, every non-trivial elt is hyperbolic:

if elliptic of finite order → cone pt.

if infinite order → problem with proper discontinuity

if parabolic: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is unipotent, hence all parabolics are unipotent (unipotency is conjugacy invariant), problem with cpts.

$$\Rightarrow \text{sys}(\Gamma \backslash \mathbb{H}^2) = \min \left\{ 2 \cosh^{-1} \left(\frac{|\text{tr}(g)|}{2} \right) \mid g \in \Gamma \setminus \{0\} \right\}$$

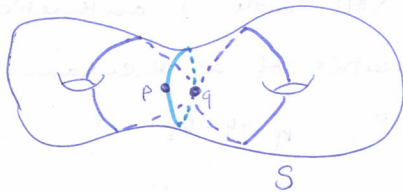
as hyperbolic is equivalent to non-trivial in Γ .

A simple upper bound on $\text{sys}(S)$

Lemma. S closed orientable hyp. surface of genus g .

$$\Rightarrow \text{sys}(S) \leq 2 \log(4g - 2).$$

Pf:



$p \in S$ point

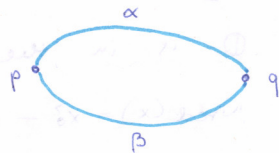
$D_r(p)$ = disc of radius r around p

Suppose $D_r(p)$ is not embedded \Rightarrow

then two radii of $D_r(p)$ meet.

Claim: the curve they form is non-contractible.

Pf: $\exists \rightarrow$ get 2 geodesics in \mathbb{H}^2 by lifting α and β which intersect twice. \downarrow



\Rightarrow we found a closed geodesic of length $\leq 2r$.

\nexists closed geod. of length $\leq \text{sys}(S)$ by definition.

So if $D_r(p)$ is not embedded then $2r \leq \text{sys}(S)$.

In particular, $D_{\frac{\text{sys}(S)}{2}}(p)$ must embed in S .

$$\Rightarrow \text{area} \left(D_{\frac{\text{sys}(S)}{2}}(p) \right) \leq \text{area}(S)$$

Gauss-Bonnet: the Gauss curvature K is -1 on S .

$$-\text{area}(S) = \int_S K = 2\pi \chi(S) = 4\pi(1-g)$$

$$\text{area} \left(D_{\frac{\text{sys}(S)}{2}}(p) \right) = \text{area}_{\mathbb{H}^2} \left(D_r(\tilde{p}) \right) = 2\pi(\cosh(r) - 1)$$

isometric to a disc

$D_r(\tilde{p})$ in \mathbb{H}^2

where $r = \frac{\text{sys}(S)}{2}$

$$2\pi \left(\cosh \left(\frac{\text{sys}(S)}{2} \right) - 1 \right) \leq 4\pi(g-1) \quad \text{The assertion follows.}$$

Next week: (Buser - Samal) \exists sequence S_ε of arithmetic hyp surfaces

of genus g_ε s.t. $g_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$ and $\text{sys}(S_\varepsilon) \geq \frac{4}{3} \log(g_\varepsilon)$.

19.06.2018

Recall the following:

Prop. There are 1-to-1 correspondences

$$\left\{ \begin{array}{l} \text{closed geodesics} \\ \text{on } S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{free lift. classes of typically} \\ \text{nontrivial closed curves on } S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{conjugacy classes} \\ \text{of } \Gamma \setminus \{e\} \end{array} \right\}$$

Moreover for a closed geodesic γ we have $l(\gamma) = 2 \cosh^{-1} \left(\frac{|\text{tr} \rho|}{2} \right)$.

Today we will construct arithmetic surfaces (i.e. $\Gamma \backslash \mathbb{H}^2$ with Γ arithmetic)

with large systoles. Reference: "On the period matrix of a Riemann surface of large genus" by Buser, P. and Samal, P., pp 44-45

Goal: find surfaces S_{g_ε} with systole $\geq \frac{4}{3} \log(g_\varepsilon)$ with genus g_ε .

Recall: $\mathbb{H}_{\mathbb{Q}}^{a,b} = \{x_0 + ix_1 + jx_2 + kx_3 \mid x_i \in \mathbb{Q}\}$ $i^2 = a, j^2 = b, ij = -ji = k$

\mathbb{Q} is in the center of $\mathbb{H}_{\mathbb{Q}}^{a,b}$

(*) $N_{\text{red}}(x) = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$. If this is $\neq 0$ for any nontrivial $\Rightarrow \mathbb{H}_{\mathbb{Q}}^{a,b}$ div. alg.

Def. $\tilde{\Gamma} := \underline{SL(1, \mathbb{H}_{\mathbb{Z}}^{a,b})} := \{g \in \mathbb{H}_{\mathbb{Z}}^{a,b} \mid N_{\text{red}}(g) = 1\}$

This is ok if a, b are as in (*).

Def. p odd prime

$\tilde{\Gamma}(p) := \{g \in \underline{SL(1, \mathbb{H}_{\mathbb{Z}}^{a,b})} \mid g \equiv 1 \pmod{p}\}$ p^{th} principal congruence subgroup of $\tilde{\Gamma}$

$$x_0 \equiv 1 \pmod{p},$$

$$x_1 \equiv x_2 \equiv x_3 \equiv 0 \pmod{p} \quad \text{for } g = x_0 + x_1 i + x_2 j + x_3 k$$

$$\tilde{\Gamma}, \tilde{\Gamma}(p) \hookrightarrow SL(2, \mathbb{R}) \twoheadrightarrow PSL(2, \mathbb{R}) = SL(2, \mathbb{R}) / \pm \text{Id}$$

$$x_0 + x_1 i + x_2 j + x_3 k \mapsto \begin{pmatrix} x_0 + x_1 \sqrt{a} & x_2 + x_3 \sqrt{a} \\ b(x_2 - x_3 \sqrt{a}) & x_0 - x_1 \sqrt{a} \end{pmatrix}$$

Lemma 1. $\tilde{\Gamma}(p)$ is torsion free, i.e. $\nexists \gamma \in \tilde{\Gamma}(p) \setminus \{1\}, n \in \mathbb{N}_{>0} \gamma^n = 1$.

Pf: Case 1: $p \nmid n, \gamma \in \tilde{\Gamma}(p) \setminus \{1\}$

$$\Rightarrow \gamma = 1 + y_0 p^{d_0} + y_1 p^{d_1} i + y_2 p^{d_2} j + y_3 p^{d_3} k, \quad p \nmid y_0, y_1, y_2, y_3,$$

not all y_r are 0 and $\forall d_r > 0$.

We will show $\gamma^n \neq 1 \pmod{p^{d+1}}$ where $d = \min \{dr \mid \gamma r \neq 0, r=1,2,3\} \geq 1$

$$\gamma^n = 1 + n(\gamma_0 p^{d_0} + \gamma_1 p^{d_1} i + \gamma_2 p^{d_2} j + \gamma_3 p^{d_3} k) + \text{higher order terms, all divisible by } p^{d+1}$$

$\neq 1 \pmod{p^{d+1}}$ because $p \nmid n$.

Case 2. $p \mid n$.

Write $\gamma^n = \gamma^{p^2 n'}$, $p \nmid n'$.

By Case 1: $\gamma^{p^2} \neq 1$. $\gamma^{p^2} = \underbrace{(\gamma^p)^p}_{s \text{ times}} \dots$

Hence it suffices to show $\gamma^p \neq 1$.

$$p^{d+1} \mid \binom{p}{i} p^{di} \quad \forall i > 1, \text{ binomial term.}$$

$$\Rightarrow \gamma^p \equiv 1 + p(\gamma_0 p^{d_0} + \dots + \gamma_3 p^{d_3} k) \pmod{p^{d+2}}$$

Cor. $\Gamma(p) \backslash \mathbb{H}^2$ is a closed hypersurface.

Lemma 2. $\tilde{\Gamma}(p) / \tilde{\Gamma}(p) = \text{SL}(2, \mathbb{Z}/p\mathbb{Z})$

PF: We have a map $\mathbb{H}_{\mathbb{Z}}^{a,b} \rightarrow \mathbb{H}_{\mathbb{Z}/p\mathbb{Z}}^{a,b}$ (reducing mod p)

which restricts to a map $\text{SL}(2, \mathbb{H}_{\mathbb{Z}}^{a,b}) \rightarrow \text{SL}(2, \mathbb{H}_{\mathbb{Z}/p\mathbb{Z}}^{a,b})$

The img of the latter is $\text{SL}(2, \mathbb{Z}/p\mathbb{Z})$ and the kernel is $\tilde{\Gamma}(p)$ (using $a,b \neq 0 \pmod{p}$)

Lemma 3. $\#\text{SL}(2, \mathbb{Z}/p\mathbb{Z}) = p(p+1)(p-1)$

PF: Start with $\#\text{GL}(2, \mathbb{Z}/p\mathbb{Z})$

$$\text{GL}(2, \mathbb{Z}/p\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}/p\mathbb{Z}, ad - bc \neq 0 \right\}$$

If $a=0$ or $d=0 \rightarrow b, c \neq 0$

$$\underbrace{(2p-1)}_{a \text{ and } d} \cdot \underbrace{(p-1)^2}_{b \text{ and } c \text{ can be chosen freely}}$$

If $ad \neq 0$

$$\rightarrow \underbrace{(p-1)^2}_{a, d \text{ chosen freely}} \cdot \left(p + \underbrace{(p-1)^2}_{\substack{\uparrow \\ \text{if } b=0 \\ \text{then } c \neq \frac{ad}{b}}} \right)$$

$$\left. \begin{array}{l} \text{If } a=0 \text{ or } d=0 \\ \text{If } ad \neq 0 \end{array} \right\} \# \text{GL}(2, \mathbb{Z}/p\mathbb{Z}) = (p-1)^2 p (p+1)$$

$$\det: GL(2, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathbb{Z}/p\mathbb{Z} \setminus \{0\}, \quad SL(2, \mathbb{Z}/p\mathbb{Z}) = \ker(\det)$$

$$\Rightarrow \#SL(2, \mathbb{Z}/p\mathbb{Z}) = \frac{\#GL(2, \mathbb{Z}/p\mathbb{Z})}{\#(\mathbb{Z}/p\mathbb{Z})^\times} = (p-1)p(p+1)$$

Lemma 4. $\exists A > 0$ s.t. $\forall p > a, b$ prime:

$$g_p = \text{genus}(\Gamma(p) \backslash \mathbb{H}^2) = p(p-1)(p+1)A + 1$$

PF: Set $A' = \text{area}(\Gamma \backslash \mathbb{H}^2)$ (There is a formula for this by Shimizu and Borel.)

$$\text{area}(\Gamma(p) \backslash \mathbb{H}^2) = \underbrace{[\Gamma: \Gamma(p)]}_{\text{covering space}} \cdot \underbrace{\text{area}(\Gamma \backslash \mathbb{H}^2)}_{A'} = p(p+1)(p-1)A' \cdot \left(\frac{1}{2}\right)$$

$\Gamma/\Gamma(p) \cong \text{PSL}(2, \mathbb{Z}/p\mathbb{Z})$

(This is not quite precise since $\Gamma \backslash \mathbb{H}^2$ may not be a manifold.)

$$\text{area}(\Gamma(p) \backslash \mathbb{H}^2) = \int_{\Gamma(p) \backslash \mathbb{H}^2} K = 2\pi \chi(\Gamma(p) \backslash \mathbb{H}^2) = 4\pi(1-g_p) \quad \text{by Gauss-Bonnet}$$

Lemma 5. $\text{sys}(\Gamma(p) \backslash \mathbb{H}^2) \geq 2 \cosh^{-1}(p^2 - 1)$

PF: Recall $\ell(\gamma) = 2 \cosh^{-1}\left(\frac{|\text{tr}(\gamma)|}{2}\right)$ for a closed geodesic γ .

Suppose $x_0 + x_1 i + x_2 j + x_3 k \in \tilde{\Gamma}(p) \rightarrow p \mid x_1, x_2, x_3$

and $1 = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2 \Rightarrow x_0^2 \equiv 1 \pmod{p^2}$

$\Rightarrow x_0 \equiv \pm 1 \pmod{p^2}$ since $(\pm 1 + mp)^2 = 1 \pm 2mp + m^2 p^2$

$$|\text{tr}(\gamma)| = \left| \text{tr} \begin{pmatrix} x_0 + x_1 \sqrt{a} & x_2 + x_3 \sqrt{a} \\ b(x_2 - x_3 \sqrt{a}) & x_0 - x_1 \sqrt{a} \end{pmatrix} \right| = |2x_0| \geq 2p^2 - 2 \quad \text{by } \Gamma(p) \backslash \mathbb{H}^2 \text{ being opt and the following exc.}$$

Exc. If A is a 2×2 matrix with $\det A = 1$, $\text{tr} A = 2 \Rightarrow A$ is unipotent

Cor. $\exists C > 0$ constant s.t. $\text{sys}(\Gamma(p) \backslash \mathbb{H}^2) \geq \frac{4}{3} \log(g_p) - C \quad \forall p > a, b$ prime

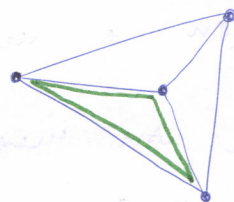
PF: $g_p \leq A'' p^3 \Rightarrow \text{sys} \geq 2 \log p^2$

$\Rightarrow \text{sys}(\Gamma(p) \backslash \mathbb{H}^2) \geq 2 \log \left(g_p^{1/2} A''' \right) = \frac{4}{3} \log(g_p) - 2 \log(A''')$

Thm. (Mumford) $\sup_{PO} \{ \text{sys}(S) \mid S \text{ closed hyp. surface of genus } g \}$ is realized.

Def. G graph is called k -regular if $\deg(v) = k \quad \forall v$ vertex of G .

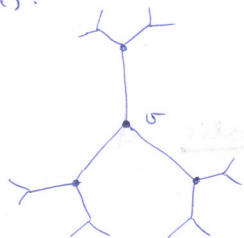
Def. girth $(G) :=$ shortest cycle in G



Lemma. G a k -regular graph on n vertices.

$$\Rightarrow \text{girth}(G) \lesssim 2 \log_{k-1}(n).$$

PF (SKETCH):



Take the "disc" around the vertex, count the vertices, they cannot coincide.

Thm. (Lubotzky - Molinier - Samotij) p odd prime $\Rightarrow \exists (p+1)$ -regular graph G with $\text{girth}(G) \gtrsim \frac{4}{3} \log_p(\# \text{vert } G)$

21.06.2018

Lattices in $SO(1, n)$

Recall that $(SO(1, n) / \pm Id)^0 = PO(1, n)^0 = \text{Isom}^+(\mathbb{H}^n)$

where $\mathbb{H}^n = \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 - x_1^2 - \dots - x_n^2 = -1, x_0 > 0 \}$

Prop. Let $a_1, \dots, a_n \in \mathbb{Z}_{>0}$, $G = SO(x_0^2 - a_1 x_1^2 - \dots - a_n x_n^2; \mathbb{R}) \cong SO(1, n)$.

If $n \geq 4$ then $G_{\mathbb{Z}}$ is an arithmetic lattice that is not compact.

PF (SKETCH): G def over $\mathbb{Q} \Rightarrow G_{\mathbb{Z}}$ is an arithl subgroup.

Meyer's thm: $\forall n \geq 4, a_i \in \mathbb{Q}: x_0^2 - a_1 x_1^2 - \dots - a_n x_n^2 = 0$ has a nontrivial integral solution.

The same argument implies the compactness of $G_{\mathbb{Z}}$.

Prop: If $n \notin \{3, 7\}$ then up to conjugation and commensurability these are all the non-compact arithmetic groups in $SO(1, n)$.

Prop: Let $F \neq \mathbb{Q}$ be a totally real algebraic number field, O_F the ring of integers, $a_1, \dots, a_n \in F^+$ s.t. $\forall \sigma \neq \text{id}$ place $\sigma(a_i) < 0 \quad \forall i = 1, \dots, n$, $G = SO(x_0^2 - a_1 x_1^2 - \dots - a_n x_n^2; \mathbb{R}) \cong SO(1, n) \Rightarrow G_{O_F}$ is a ccopt ar. subgroup of G .

PF: Same as in 2-dim case.

Prop. For $2|n$ there are up to conjugacy and commensurability all the coapt with subgrps of $SO(1,n)$.

For $2 \nmid n$ we need a second construction using quaternion algebras.

Question. Are there any non-arithmetic lattices?

Yes \rightarrow Gromov - Radošević - Shapiro

Construction of non-arithmetic lattices $n \geq 3$

Def. A connected Riemannian mf. M is called hyperbolic if

- 1) its universal cover \tilde{M} is isometric to \mathbb{H}^n
- 2) M is orientable
- 3) M is compact (is this implied by 1?)

Prop. A connected Riemannian mf. M of finite volume is hyperbolic iff $\exists T < SO(1,n)^\circ$ torsion-free lattice s.t. M is isometric to $\Gamma \backslash \mathbb{H}^n$.

Idea. Build hybrid manifolds: not arithmetic but built up from arithmetic pieces.

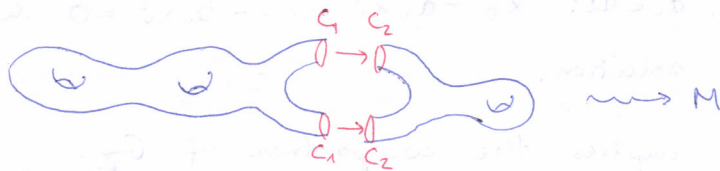
Hybrid manifolds



C_1, C_2 totally geodesic, \exists isometry $C_1 \rightarrow C_2$

Γ_1, Γ_2 not commensurable

Cut and glue:



This works b/c we have codim 1.

Prop. Suppose

- M_1, M_2 are connected mfs w/o bdrng,
- $f: \partial M_1 \rightarrow \partial M_2$ a homeo.

Then define $M_1 \cup_f M_2$; this is an n -dim conn. mf. w/o bdrng. \square

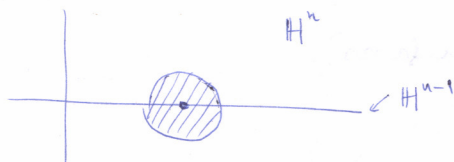
This means that topologically our construction works; the problem is if it works geometrically.

Problem. Gluing 2 Riemannian mfs along bdy does not always give a Riemannian mf, even if the gluing map is an isometry.

Ex. $D_1, D_2 \cong_{\text{isom}} \{z \in \mathbb{R}^2 \mid \|z\|=1\}$, $f: \partial D_1 \rightarrow \partial D_2$

If we could extend the metric to $D_1 \cup_f D_2$ then we would get a flat metric on S^2 which cannot occur by Gauss-Bonnet.

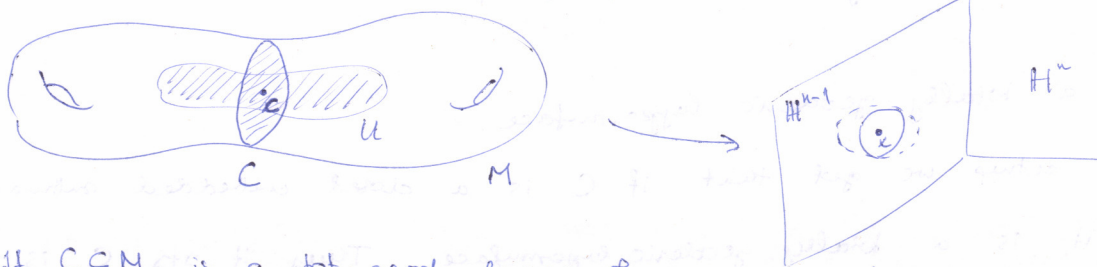
So when is this possible? Subtively things have to look like:



This means the following formally:

Def. M a hyperbolic mf. A totally geodesic hypersurface in M is a closed non-empty connected submf. $C \subseteq M$ s.t. $\forall c \in C$:

- $\exists U \subseteq M$ open nbh. of c
- $\exists x \in \mathbb{H}^{n-1} \subseteq \mathbb{H}^n$
- $\exists V \subseteq \mathbb{H}^n$ nbhd of x
- $\exists g: U \rightarrow V$ Riem isometry s.t. $g(c) = x$, $g(U \cap C) = \mathbb{H}^{n-1} \cap V$.



Prop. If $C \subseteq M$ is a tot. geod. hypersurface in a hyp. n -mf. of finite volume, then there are

- $\Gamma < PO(n, 1)$ lattice
- $f: M \rightarrow \Gamma \backslash \mathbb{H}^n$ isometry s.t. $f(C)$ is the img of \mathbb{H}^{n-1} in $\Gamma \backslash \mathbb{H}^n$

In particular, C is a hyperbolic $(n-1)$ -mf.

Prop. If M_1, M_2 are hyp. n -manifolds of finite volume,

• $C_j \subseteq M_j$ totally geodesic hypersurfaces s.t. $M_j \setminus C_j$ is connected

• $f: C_1 \rightarrow C_2$ a Riemannian isometry

then $\overline{M_1 \setminus C_1} \cup_{f,f} \overline{M_2 \setminus C_2}$ is a hyp. mf. where $\overline{M_j \setminus C_j}$ is $M_j \setminus C_j$ without two copies of C_j attached as boundary and $\cup_{f,f}$ means that we use two copies of f for gluing.

Pf. Picture + EXC. 6.5.4. □

How do we obtain totally geodesic hypersurfaces?

Lemma. Suppose we have

• $\Gamma < PO(1, n)^0$ torsion free lattice

• C the img of H^{n-1} in $\Gamma \backslash H^n$

• $\tau: H^n \rightarrow H^n$ reflection across H^{n-1} , i.e. $\tau(x_0, \dots, x_{n-1}, x_n) = (x_0, \dots, x_{n-1}, -x_n)$

(conversely, the isometry of H^{n-1} can be extended to perenne isometry of H^n (?))

such that the following hold:

• $\Gamma \cap PO(1, n-1)$ is a lattice

• $\exists \Gamma' < PO(1, n)^0$ torsion-free lattice s.t.

• Γ' is normalised by τ , i.e. $\tau \Gamma' \tau = \Gamma'$

• $\Gamma < \Gamma'$

Then C is a totally geodesic hypersurface.

Pf. From the setup we get that if C is a closed embedded submanifold then it is a totally geodesic hypersurface. Thus it sts C is a closed embedded submf.

$\Gamma_0 := \{ \gamma \in \Gamma \mid \gamma H^{n-1} = H^{n-1} \} \Rightarrow \Gamma \cap PO(1, n-1) < \Gamma_0$ is of index ≤ 2
 $\Rightarrow \Gamma_0$ is a lattice.

By EXC. 4.4.3 we get a map $\psi: \Gamma_0 \backslash H^n \rightarrow \Gamma \backslash H^n$ a proper embedding.

(Proper: preimg of cpt is cpt. If $f: X \rightarrow Y$ is proper, Y loc. cpt. T2 $\Rightarrow f$ closed.)

We need to show injectivity.

Suppose $\gamma x = y$ for some $\gamma \in \Gamma$, $x, y \in \mathbb{H}^{n-1}$

WTS: $\gamma \in \Gamma \cap \text{PO}(n,1)$

We have $\gamma^{-1} \tau \gamma \in \Gamma'$ and $\gamma^{-1} \tau \gamma(x) = x$

Γ' is torsion-free \Rightarrow no elt of $\Gamma' \setminus \{e\}$ has a fixed pt. in \mathbb{H}^n

$\Rightarrow \gamma^{-1} \tau \gamma = e \Rightarrow \gamma$ commutes with $\tau \xrightarrow{\text{Exc.}} \gamma$ preserves the fixed points of $\tau \Rightarrow \gamma$ preserves $\mathbb{H}^{n-1} \Rightarrow \gamma \in \Gamma \cap \text{PO}(1, n-1)$

Thm. Suppose we have

26.06.2018

- M_1, M_2 hyperbolic mfs of finite volume
- $G_j \subseteq M_j$ totally geodesic Riemannian hypersurface
- $f: C_1 \rightarrow C_2$ Riem. isometry
- C_1, C_2 have finite volume as hyperbolic $(n-1)$ -mfs.
- Both $M_1 \setminus C_1$ and $M_2 \setminus C_2$ are connected.

Then if $\overline{M_1 \setminus C_1} \cup_{f, f^{-1}} \overline{M_2 \setminus C_2}$ is arithmetic then M_1 is commensurable to M_2 .

Cor. $\forall n \geq 3$ odd there exist non-copt non-arithmetic lattices in $\text{PO}(1, n)$

(The assumptions $2 \nmid n$ and non-coptness are unnecessary, see the EXERCISES in § 6.5.)

Prop. $B_1, B_2: \mathbb{Q}^{n+1} \rightarrow \mathbb{Q}$, $B_1(x) := x_0^2 - x_1^2 - \dots - x_n^2$

$$B_2(x) := x_0^2 - x_1^2 - \dots - x_{n-1}^2 - 2x_n^2$$

Let $\Gamma_1 := \text{SO}(B_1, \mathbb{Z})$, $\Gamma_2 := \text{SO}(B_2, \mathbb{Z}) h^{-1}$ where $h := \text{diag}(1, \dots, 1, \sqrt{2})$

(Thus both $\Gamma_1, \Gamma_2 < \text{SO}(1, n)$.) Let $M_j := \Gamma_j \backslash \mathbb{H}^n$, $C_j := \text{img of } \mathbb{H}^{n-1} \text{ in } M_j$,

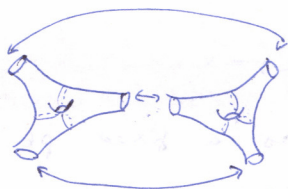
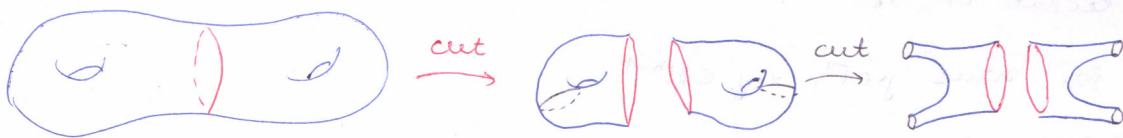
$$\hat{\Gamma}_j := \Gamma_j \cap \text{SO}(1, n-1)$$

- Issues:
- are $M_j \setminus C_j$ connected?
 - are $\Gamma_1 \approx \Gamma_2$ commensurable?

Step 1. Pass to finite index subgps s.t. $\hat{\Gamma}_j$ are torsion free

Both $\text{SO}(B_j, \mathbb{Z})$ are normalised by τ , hence G_j tot. geod. by Lemma

Claim. By passing to finite index subgroups wma $M_j \setminus C_j$ to be connected.



Goal: Imitate this approach

Find finite cover $\tilde{M}_j \xrightarrow{\pi} M_j$

s.t. $\pi^{-1}(C_j)$ has multiple conn. components.

Pf of CLAIM: Let $R_j := \mathbb{Z}[\text{matrix entries of } \Gamma_j] \subseteq \mathbb{R}$

This is finitely generated because Γ_j is (this follows from a thm. we didn't do in class.) R_j is an integral domain with 1

$\forall x \in R_j \setminus \{0\} \exists I_j \subseteq R_j$ ^{max.} ideal s.t. $x \notin I_j$ (This is true because \mathbb{Z} is a Jacobson ring, every finitely generated ring over a Jacobson ring is Jacobson itself, so in particular, R_j is Jacobson, i.e. the intersection of all maximal ideals is $\{0\}$. Note that finiteness is essential: $0 \neq p \in \mathbb{Z}_p$ is contained in the unique max. ideal $p\mathbb{Z}_p$.)

Let $g \in \Gamma_j \setminus \hat{\Gamma}_j \Rightarrow \exists$ matrix entry g_{kl} s.t. $\tau(g)_{kl} \neq g_{kl}$

$\Rightarrow x := \tau(g)_{kl} - g_{kl} \in R_j \setminus \{0\}$. Let I_j be a max. ideal s.t. $x \notin I_j$.

Consider the map

$$\begin{aligned} \Gamma_j < SO(1, n)_{R_j} &\longrightarrow SO(1, n)_{R_j/I_j} \times SO(1, n)_{R_j/I_j} \\ \gamma &\longmapsto ([\gamma]_{I_j}, [\tau(\gamma)]_{I_j}) \end{aligned}$$

Note that $\hat{\Gamma}_j$ lands in the diagonal while Γ_j does not (by construction).

We obtain a finite quotient of Γ_j in which $\hat{\Gamma}_j$ maps to a smaller finite group. \Rightarrow The finite cover of M_j corresponding to this quotient has multiple copies of $C_j \Rightarrow$ one of them will no longer separate

Thus we have $\Gamma_1 \cong \Gamma_2$. Exc. 6.5.9: we can pass to further finite cover to make C_1 and C_2 isometric. □

We can perform gluing: get $M = \overline{M_1 \setminus C_1} \cup_{\text{fit}} \overline{M_2 \setminus C_2}$ (noncompact)

Goal: show that M is non-arithmetic, i.e. show $M_1 \not\sim_{\text{comm.}} M_2$.

Step 1. We can recover the quadratic form from Γ_1 .

PF: Ex. 6.4.2: $G < GL(\ell, \mathbb{C})$ Lie group that does not fix a proper nonzero subspace of \mathbb{C}^ℓ and B_1, B_2 are G -invariant quadratic forms then $\exists \lambda \in \mathbb{C}$ s.t. $B_1 = \lambda B_2$.

Γ lattice + Borel density \Rightarrow quad. form fixed by Γ is unique up to multiple.

Suppose $\exists g \in O(1, n)$ s.t. $g\Gamma_2 g^{-1}$ commensurable to Γ_1

$\Rightarrow g' = gh$ has $(g')^t \text{diag}(-1, 1, \dots, 1, 2) g' = \lambda \text{diag}(-1, 1, \dots, 1)$ where $\lambda \in \mathbb{Q} \setminus \{0\}$.

Consider the discriminant of B_j def (B_j)

$$\text{disc}(B_1) = -1, \text{disc}(B_2) = -2$$

∇ Commensurability $\Rightarrow -c\lambda^{n+1} = -1 \Rightarrow 2$ is a square in $\mathbb{Q} \setminus \mathbb{Z}$

Hence M is non-arithmetic.

PF OF COMMENSURABILITY THM:

- Lemma. If
- G has no cpt factors
 - Γ_1, Γ_2 arithmetic lattices in G
 - $\Gamma_1 \cap \Gamma_2$ Zariski dense in G

then Γ_1, Γ_2 are commensurable.

PF OF LEMMA: By def we have isomorphisms $\varphi_j: G \xrightarrow{\sim} H_j$ s.t. $\varphi_j(\Gamma_j) = (H_j)_{\mathbb{Z}}$

Define $\varphi: G \rightarrow H_1 \times H_2$ i.e. $\varphi = \varphi_1 \times \varphi_2$.

$$g \mapsto (\varphi_1(g), \varphi_2(g))$$

$\varphi(G)_{\mathbb{Z}} = \varphi(\Gamma_1 \cap \Gamma_2)$ Zariski dense in $\varphi(G)$

Prop. 5.1.8 $\Rightarrow \varphi(G)$ def'd over $\mathbb{Q} \Rightarrow \varphi(G)_{\mathbb{Z}}$ is a lattice

$\Rightarrow \Gamma_1 \cap \Gamma_2 < \Gamma_1$ is of finite index
 $< \Gamma_2$

Thm. (mild generalisation of what was stated before)

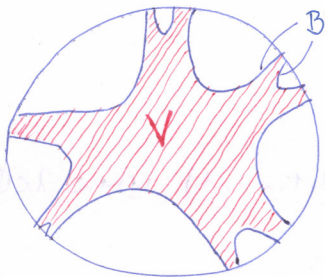
- M_1, M_2 n -manifolds
- $M'_j \subset M_j$ submf. with non-empty totally geodesic boundary
- $f: \partial M'_1 \rightarrow \partial M'_2$ isometry
- $\partial M'_j$ has finitely many conn. components, all of which have finite volume

If $M = M'_1 \cup_f M'_2$ is arithmetic then $M \approx M_f$. Change "manifold" to "arithmetic hyperbolic manifold".

Pf. Setup:
 $M = \Gamma \backslash \mathbb{H}^n$, $\varphi: \mathbb{H}^n \rightarrow M$ covering map

$$B := \varphi^{-1}(\partial M'_1) \subseteq \mathbb{H}^n$$

$V :=$ closure of a component of $\mathbb{H}^n \setminus B$ containing a point from $\varphi^{-1}(M'_1)$



$\dim = 2$

$$\Gamma' := \{ \gamma \in \Gamma \mid \gamma V = V \}$$

Exc. 6.5.12

$$= \{ \gamma \in \Gamma \mid \text{int}(\gamma V \cap V) = \emptyset \}$$

Since $M'_1 = \varphi(V)$ we get $M'_1 = \Gamma' \backslash V$

Def. $U \subseteq \mathbb{H}^n$ is convex if $\forall x, y \in U$ the geodesic segment between x and y is contained in U .

V is the intersection of half-spaces $\Rightarrow V$ is convex $\Rightarrow V$ is simply connected $\Rightarrow V$ is the universal cover of M'_1 .

Similarly, write $M_1 = \Gamma_1 \backslash \mathbb{H}^n$, φ_1 the covering map, B_1 the lift of the boundary of M_1 , V_1 the univ. cover of M_1 , Γ'_1 the corresponding subgroup.

By the uniqueness of the universal cover there is a Riemannian isometry $V \rightarrow V_1$.

We can move every totally geodesic copy of \mathbb{H}^{n-1} in \mathbb{H}^n to "the standard" copy with an isometry of \mathbb{H}^n , where "the standard" copy is

$$\mathbb{H}^{n-1} = \{ (x_0, \dots, x_n) \in \mathbb{R}^n \mid x_0 > 0, x_n = 0, x_0^2 - x_1^2 - \dots - x_{n-1}^2 = -1 \}$$

Thus we may set everything up in such a way that the standard \mathbb{H}^{n-1} is one of the ∂ -components of V and the isometry $V_1 \rightarrow V$ is the identity.

$$\Rightarrow \Gamma'_1 = \Gamma' = \Gamma \cap \Gamma_1$$

Recall the following:

Lemma. If G has no cpt factors,

- Γ_1, Γ_2 are arith. lattices in G
- $\Gamma_1 \cap \Gamma_2$ is Zariski dense in Γ_1

then Γ_1 and Γ_2 are commensurable.

Goal: the Zariski closure of Γ' contains $PO(n, 1)^\circ$.

Then it follows from the lemma that $\Gamma_1 \approx \Gamma$ and hence $M_1 \approx M$.

\mathbb{F} : ∂M_1 fin. vol. $\Rightarrow \Gamma' \cap SO(1, n-1)$ is a lattice

Borel density thm. $\Rightarrow \bar{\Gamma}'$ is the Zariski closure of Γ' , $\bar{\Gamma}'^\circ \supseteq PO(1, n)^\circ$

Fact. (Exc. 6.5.13) $PO(1, n-1)^\circ$ is a maximal connected subgroup of $PO(1, n)^\circ$

So if $\bar{\Gamma}'^\circ \not\supseteq PO(1, n-1)^\circ$ then $\bar{\Gamma}'^\circ = PO(1, n)^\circ$

Suppose $\bar{\Gamma}'^\circ = PO(1, n-1)^\circ$. Since $\bar{\Gamma}'^\circ < \bar{\Gamma}'$ has finite index, $PO(1, n-1)^\circ$ contains a finite index subgp of $\bar{\Gamma}'$.

In fact, $\{\gamma \in \Gamma' \mid \gamma H = H\}$ has finite index in Γ' $\forall H \subseteq \partial V$ connected component.

Claim. This leads to a contradiction.

Case 1. ∂V is connected

Up to finite index: $\Gamma' < PO(1, n-1)$

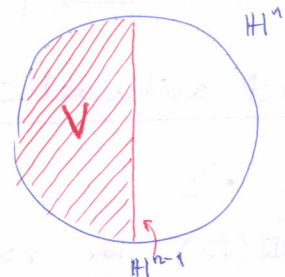
Let $\tau: \mathbb{H}^n \longrightarrow \mathbb{H}^n$
 $(x_0, \dots, x_n) \longmapsto (x_0, \dots, x_{n-1}, -x_n)$

τ centralises Γ'

$\mathbb{H}^n = V \cup \tau(V)$

$\Gamma' \backslash V$ and $\Gamma' \backslash \tau(V)$ are fin. vol hyp mfs. $\Rightarrow \Gamma' \backslash \mathbb{H}^n$ is a fin. vol hyp mf.

$\Rightarrow \Gamma'$ is a lattice in $PO(1, n)$, contradicting Borel density \Leftarrow



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \tau$$

Case 2a. ∂V not connected, $\partial M_1'$ cpt.

H_1, H_2 two distinct conn. components of ∂V

$\Gamma' \backslash H_1$ is compact $\Rightarrow \exists$ compact subset $C \subseteq H_1$ s.t. $\Gamma' C = H_1$

$$\delta := \min \{ \text{dist}(c, H_2) \mid c \in C \} > 0$$

Since Γ' acts by isometries, $\delta = \text{dist}(H_1, H_2)$.

Fact. (negative curvature) $\exists! p \in H_1$ s.t. $\text{dist}(p, H_2) = \delta$.

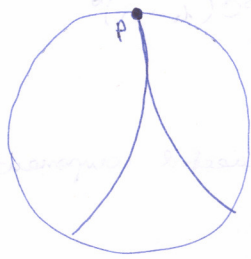
$\Rightarrow \forall \gamma \in \Gamma'$ fixes p , and since Γ' acts by isometries, $\Gamma' = \{e\}$ \swarrow \uparrow
and Γ' acts freely

Something is fishy, infinite indices may be at play.

Case 2b. ∂V not connected, $\partial M_1'$ non-cpt.

The proof is similar: if H_1, H_2 have positive distance then the argument from 2a works.

When no such pair of conn. components exist.



Hyperbolic geometry \Rightarrow they must intersect at a single point $p \in \partial_\infty H^n$

This is a fixed point.

Exc. $\Rightarrow \Gamma'$ is not Zariski dense. \swarrow This finishes the pf of the thm. \square

When $H_2 = H^n$, and since $PO(1, n-1)^0$ contains a finite index subgroup of Γ' things will work out.

Noncompact arithmetic subgroups of $SL(3, \mathbb{R})$

Prop. Let

• $L := \mathbb{Q}(\sqrt{r})$ for $r \geq 2$ square-free integer

• $\sigma \neq \text{id}$ place of L

$$\bullet \mathfrak{F}_3 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\bullet \Gamma = \text{SU}(\mathfrak{F}_3, \sigma, \mathbb{Z}[\sqrt{r}]) = \{ g \in \text{SL}(3, \mathbb{Z}[\sqrt{r}]) \mid \sigma(g^T) \mathfrak{F}_3 g = \mathfrak{F}_3 \}$$

Then 1) Γ is an arith. subgroup of $SL(3, \mathbb{R})$

2) Γ is not cocompact

3) Γ is not commensurable to $SL(3, \mathbb{Z})$.

$$\text{Pf: } \Delta: L^3 \rightarrow \mathbb{R}^6, \quad \Delta(v) = (v, \mathcal{F}_3(v))$$

$$V_{\mathbb{Q}} := \Delta(L^3)$$

$$L := \mathbb{Z}[\sqrt{r}]^3$$

$$\rho: SL(3, \mathbb{R}) \rightarrow SL(6, \mathbb{R}) \text{ defined by } \rho(A)(v, w) = (Av, (A^T)^{-1}w)$$

$$\forall v, w \in \mathbb{R}^3$$

- Then
- $V_{\mathbb{Q}}$ is a \mathbb{Q} -form for \mathbb{R}^6
 - L is a \mathbb{Z} -lattice for $V_{\mathbb{Q}}$
 - ρ is a homomorphism
 - $\Gamma = \{g \in SL(3, \mathbb{R}) \mid \rho(g)L = L\}$

To prove 1) we nts that $\rho(SL(3, \mathbb{R}))$ is def. over \mathbb{Q} .

Recall: nts $\rho(SL(3, \mathbb{R}))_{\mathbb{Q}}$ is dense in $\rho(SL(3, \mathbb{R}))$.

$$\text{Let } U_1 := \left\{ \begin{pmatrix} 1 & a+b\sqrt{r} & \frac{-a^2-rb^2}{2} + c\sqrt{r} \\ 0 & 1 & -a+b\sqrt{r} \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Q} \right\} \quad \text{i.e. we work with unipotents in extreme explicitly}$$

$$\rho(U_1) \subseteq \rho(SL(3, \mathbb{R}))_{\mathbb{Q}} \text{ and } U_1 \text{ is dense in } U := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$\text{likewise we may construct } U_2 \in U^T = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} \right\} \text{ dense.}$$

$$\langle U, U^T \rangle = SL(3, \mathbb{R}) \quad \checkmark$$

2): elts in U_{Γ} are contained in $\rho(\Gamma)$ and unipotent.

Godement \Rightarrow non-coapt. \checkmark

3): Sketch: • U_{Γ} maximal unipotent subgroup of Γ .

$$\text{choose } w \in \mathbb{Z}[\sqrt{r}] \setminus \{0\} \text{ s.t. } \sigma(w) = \frac{1}{w}$$

$$h := \text{diag}(w, 1, w^{-1}) \text{ satisfies } hU_{\Gamma}h^{-1} = U_{\Gamma} \text{ (just computational)}$$

$$\Rightarrow U_{\Gamma} \text{ has infinite index in } N_{\Gamma}(U_{\Gamma}).$$

• $U_{\mathbb{Z}}$ mac. unipotent subgp of $SL(3, \mathbb{Z})$

$$\text{If } U'_{\mathbb{Z}} < SL(3, \mathbb{Z}) \text{ commensurable to } U_{\mathbb{Z}} \text{ then } [N_{SL(3, \mathbb{Z})}(U'_{\mathbb{Z}}), U'_{\mathbb{Z}}] < \infty$$

and all maximal unipotent subgroups of $SL(3, \mathbb{Z})$ are conjugate.

Three main ingredients:

- (a) Siegel sets (definition)
- (b) Prove that they have finite measure
- (c) Siegel sets are coarse fundamental domains for $SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R})$, i.e. larger than an actual fund. domain but still sufficient for our purposes

This is called reduction theory. The goal is to make every elt. in $SL(n, \mathbb{R})$ as simple as possible using $SL(n, \mathbb{Z})$.

§ 7.1. Iwasawa decomposition

Thm. (Iwasawa decomposition) $G = SL(n, \mathbb{R})$. Then $G = KAN$ where

$$K = SO(n), \quad A = \left\{ \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}^0 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & * \\ & \ddots \\ 0 & & 1 \end{pmatrix} \right\}$$

Moreover, $\forall g \in G$ has a unique representation $g = kan$ with $k \in K, a \in A, n \in N$.

Pf. Uniqueness: $k_1 a_1 n_1 = k_2 a_2 n_2 \Rightarrow k_1^{-1} k_2 = a_1 n_1 n_2^{-1} a_1^{-1} \in K \cap AN$

This follows from $AN = NA$ which is easily seen from def.

Claim: $K \cap AN = \{e\}$.

$x \in K \cap AN \Rightarrow$ all eigenvalues are ± 1 as it is in SO

But eigenvalues of an upper diag matrix are the elts of the diagonal, hence all eigenvalues are $+1$. Since $x \in SO(n)$, this implies $x = e$. ✓

$$\Rightarrow k_1 = k_2 \Rightarrow a_1^{-1} a_2 = n_1^{-1} n_2 \in A \cap N = \{e\} \quad \checkmark$$

Existence: Take e_1, \dots, e_n to be the std basis of \mathbb{R}^n .

$\forall g \in G$: $\{g e_i\}_i$ is a basis of \mathbb{R}^n . Let $v_i := g e_i$

Gram-Schmidt: $w_i^* = v_i - \sum_{j=1}^{i-1} (v_i, w_j^*) w_j^*, \quad w_i := \frac{1}{\|w_i^*\|} \cdot w_i^*$

$\{w_i\}_i$ ONB.

$$\Rightarrow \exists k \in O(n) \text{ s.t. } k^{-1} e_i = w_i \quad \forall i = 1, \dots, n$$

$$\Rightarrow \exists a \in A \text{ s.t. } k w_i^* = a e_i \quad (\text{note that the norm came back})$$

By an inductive argument we get that $w_i \in \text{span}_{\mathbb{R}}\{v_1, \dots, v_i\}$.

$$\Rightarrow g^{-1}w_i^* = g^{-1}v_i - g^{-1} \sum_{j=1}^{i-1} (v_i, w_j) w_j \in g^{-1}v_i - g^{-1} \text{span}_{\mathbb{R}}\{v_1, \dots, v_{i-1}\} \\ \xrightarrow{\text{reversing the GS step}} = \varepsilon_i - \text{span}_{\mathbb{R}}\{\varepsilon_1, \dots, \varepsilon_{i-1}\}$$

$$\Rightarrow \exists u \in N : g^{-1}w_i^* = u\varepsilon_i \quad \forall i=1, \dots, n$$

$$\Rightarrow u^{-1}g^{-1}w_i^* = \varepsilon_i = a^{-1}kw_i^*$$

$$\Rightarrow g = k^{-1}a u^{-1} \in \text{KAN}. \quad \text{The last step also gives } \det(k) = 1.$$

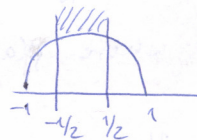
§7.2 Siegel sets

We now let go of the ass. that Γ is a lattice.

Recall (for purposes of inspiration): $SL(2, \mathbb{R}) \backslash \mathbb{H}^2 = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$

by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$. Then we have a fund domain: (not a strict fund. dom.)

$$\overline{\mathbb{F}}_0 = \left\{ z \in \mathbb{C} \mid \text{Im}(z) > 0, |z| \geq 1, -\frac{1}{2} < \text{Re } z < \frac{1}{2} \right\}$$



Lemma. $H < G$ closed subgroup, $\overline{\mathbb{F}}$ strict fund domain for $\Gamma \backslash G/H$.

Then $\forall x \in G/H : \mathbb{F} := \{g \in G \mid gx \in \overline{\mathbb{F}}\}$ is a strict fund domain for $\Gamma \backslash G$.

Recall. $\Gamma \backslash Y$ for a top space Y , $\Gamma \subset Y$ Boel, $\mathbb{F} \rightarrow Y/\Gamma$ is bijective means that \mathbb{F} is a strict fund. domain.

Proof Lemma: Exc. 7.2.1.

We obtain a fund domain $\mathbb{F}_0 = \{g \in SL(2, \mathbb{R}) \mid g(i) \in \overline{\mathbb{F}}_0\}$

Problem: this has a way too complicated shape.

Solution: build a coarse fund domain.

$$\overline{\mathbb{F}} := \{x+iy \mid c_1 \leq x \leq c_2, c_3 \leq y\} \quad \text{box}$$

Def. Let $\Gamma \backslash Y$, Y top space, properly discontinuous ^{action}. Then $\mathbb{F} \subset Y$ is called a coarse fund domain if (1) $\Gamma \mathbb{F} = Y$

and (2) $\{\gamma \in \Gamma \mid \gamma \mathbb{F} \cap \mathbb{F} \neq \emptyset\}$ is finite.

$$\text{Set } \mathbb{F} := \{g \in G \mid g(i) \in \overline{\mathbb{F}}\}, \quad \underline{N_{c_1, c_2}} := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid c_1 \leq t \leq c_2 \right\},$$

$$\underline{A_{c_3}} := \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid e^{2t} \geq c_3 \right\}, \quad K := SO(2).$$

Then $\mathbb{F} = N_{c_1, c_2} A_{c_3} K$, see Exc. 7.2.10, just computation.

Def. $G = SL(n, \mathbb{R})$. A Siegel set is a set of the form $S = \bar{N}AK$ where

$$\bar{N} = N_{c_1, c_2} = \{ n \in N \mid \forall i, j: c_1 \leq u_{ij} \leq c_2 \}$$

$$\bar{A} = A_{c_3} = \{ a \in A \mid \forall i=1, \dots, n-1: a_{ii} \geq c_3 a_{i+1, i+1} \}$$

Notation: $S_{c_1, c_2, c_3} = N_{c_1, c_2} A_{c_3} K$.

Prop. S_{c_1, c_2, c_3} has finite measure wrt the Haar measure on G .

Pf: Step 1. Description of Haar measure on G .

Write $a \in A$, $\psi_A: N \rightarrow N$
 $n \mapsto ana^{-1}$

Then if du is the Haar measure on N then

$$\int_N f(ana^{-1}) du = \int_N f(u) \rho(a) du \quad \forall f \in C_c(N).$$

where $\rho(a) = |\det(D\psi_A)|$

Uniqueness + unimodularity yield (Exc. 7.2.12):

$$\int_G f dg = \int_K \int_N \int_A f(kua) da du dk$$

where da, du, dk are the Haar measures on A, N, K resp.

$$= \int_K \int_A \int_N f(kau) \rho(a) du da dk$$

Computation (Exc. 7.2.13): $\rho(a) = \prod_{i < j} \frac{a_{jij}}{a_{iic}}$

Using that du is the Lebesgue measure in coordinates it follows (Exc. 7.2.14)

that S_{c_1, c_2, c_3} has finite measure.

§ 7.3. $\exists c_1, c_2, c_3$ for which S_{c_1, c_2, c_3} is a coarse fundamental domain

Thm. $G = SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z}) \Rightarrow G = \Gamma S_{0, 1/2}$

Lemma. $L \subseteq \mathbb{R}^n$ \mathbb{Z} -lattice. Then \exists ordered basis of \mathbb{R}^n $\{v_1, \dots, v_n\}$ s.t.

(1) $\{v_i\}$ are a generating set of L as an ab. gp.

(2) $\| \text{proj}_i^\perp v_{i+1} \| \geq \frac{1}{2} \| \text{proj}_{i-1}^\perp v_i \|$ for $i=1, \dots, n-1$ where $\text{proj}_i^\perp: \mathbb{R}^n \rightarrow V_i^\perp$

is the projection to $V_i^\perp = \text{span}_{\mathbb{R}} \{v_1, \dots, v_i\}^\perp$

PF OF LEMMA:

The proof is similar to one from the beginning of this course.

Choose $v_1 \in L \setminus \{0\}$ of minimal length.

Define v_2, \dots, v_n inductively:

given v_1, \dots, v_i choose $v_{i+1} \in L$ s.t. \bullet $\text{proj}_i^\perp v_{i+1}$ is as short as possible

\bullet v_{i+1} lin. indep. of $\{v_1, \dots, v_i\}$, i.e.

$$\text{proj}_i^\perp v_{i+1} \neq 0.$$

Goal: verify (1) and (2).

(1) $\forall i=1, \dots, n \quad L_i := \text{span}_{\mathbb{Z}} \{v_1, \dots, v_i\}$

\nexists Suppose v_1, \dots, v_n do not generate L , i.e. $L_n \neq L$.

Then $\exists i \geq 1$ minimal s.t. $L_{i+1} \neq L \cap V_{i+1}$

$$\Rightarrow \text{proj}_i^\perp L_{i+1} \not\subseteq \text{proj}_i^\perp (L \cap V_{i+1}) \quad \text{because } L \cap V_i = L_i.$$

$$\Rightarrow \exists v \in L \cap V_{i+1} \text{ with } \text{proj}_i^\perp (v_{i+1}) = k \text{proj}_i^\perp (v) \text{ for some } k \geq 2$$

because $\text{proj}_i^\perp (L \cap V_{i+1})$ is a cyclic group.

But this contradicts minimality of v_{i+1} . \nexists

(2) Wma $i=1$ for simplicity.

$$\text{Let } v_2^* := \text{proj}_1^\perp v_2. \Rightarrow \exists \alpha \in \mathbb{R}: v_2 = v_2^* + \alpha v_1$$

$$\text{Choose } k \in \mathbb{Z}: |\alpha - k| \leq \frac{1}{2}.$$

$$\begin{aligned} \nexists \text{ If } \|\text{proj}_1^\perp v_2\| < \frac{1}{2} \|v_1\| \text{ then } \|v_2 - kv_1\| &= \|v_2^* - (\alpha - k)v_1\| \\ &\leq \|v_2^*\| + |\alpha - k| \|v_1\| < \|v_1\| \end{aligned}$$

which contradicts minimality of $\|v_1\|$. \nexists

PF OF THM: $G = S_{0,1,1/2}^- \Gamma$ where $S_{0,1,1/2}^- = KA_{1/2}^- N_{0,1} \Gamma$

05.07.21

$$\text{and } A_c^- := \{a^{-1} \mid a \in A_c\} = \left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{nn} \end{pmatrix} \mid a_{ii} \leq \frac{a_{i+1,i+1}}{c} \quad \forall i=1, \dots, n-1 \right\}$$

Goal: given $g \in G$ show $\exists \gamma \in SL(n, \mathbb{Z}): g\gamma \in KA_{1/2}^- N_{0,1}$

Let $L := g\mathbb{Z}^n$ and e_1, \dots, e_n the std basis of \mathbb{R}^n .

Lemma $\Rightarrow v_1, \dots, v_n$ generating set

Claim: una $g e_i = v_i$, i.e. $\exists \gamma \in \Gamma: g \gamma e_i = v_i$

Pf: Any pair of generator sets for \mathbb{Z}^n are related by a $GL(n, \mathbb{Z})$ -

matrix: $\exists \gamma \in GL(n, \mathbb{Z}): g \gamma e_i = v_i$

By the same argument as in the pf of Mahler compactness:

Γ preserves volume hence lies in $SL(n, \mathbb{Z})$. ✓

Now use Iwasawa decomposition $g = k a u$

rotate such that $k = e \Rightarrow g = a u$, hence upper triangular with diagonal entries coming from a .

Step 1. $a \in A_{1/2}$, $a_{ii} = g_{i,i} = \left\| \text{proj}_{i-1}^{\perp} \frac{1}{g} g e_i \right\| = \left\| \text{proj}_{i-1}^{\perp} v_i \right\|$

$$\Rightarrow a_{ii} \leq \frac{1}{2} a_{i+1, i+1} \Rightarrow a \in A_{1/2}^-$$

Step 2. Using another $SL(n, \mathbb{Z})$ -matrix.

Claim: $N = N_{0,1} \mathbb{N} \mathbb{Z}$

E.g. in dim 2:

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\in \mathbb{N} \mathbb{Z}} \cdot \underbrace{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}_{\in \mathbb{N}} = \begin{pmatrix} 1 & x+1 \\ 0 & 1 \end{pmatrix}$$

Ch. 8. Real rank

Def. A closed connected subgroup $T < G$ is a torus if it is diagonalisable over \mathbb{C} i.e. $\exists g \in GL(n, \mathbb{C})$ s.t. $g T g^{-1}$ consists of diagonal matrices.

A torus is called \mathbb{R} -split if it is diagonalisable over \mathbb{R} .

Ex. 1) $A = \left\{ \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \right\}^{\circ} \subset SL(n, \mathbb{R})$ is an \mathbb{R} -split torus

2) $SO(1,1)^{\circ} \subset SL(2, \mathbb{R})$ is an \mathbb{R} -split torus (Exc. 8.1.1)

3) $SO(2) < SL(2, \mathbb{R})$ is a torus but not \mathbb{R} -split

$$\parallel \\ \{A \in SL(2, \mathbb{R}) \mid AA^T = \text{Id}\}$$

Thm. $A_1, A_2 < G$ maximal \mathbb{R} -split tori $\Rightarrow \exists g \in G: g A_1 g^{-1} = A_2$

Nb. This theorem uses the semisimplicity of G and does not hold in general.

Def. $\text{rank}_{\mathbb{R}}(G) :=$ dimension of (a) maximal \mathbb{R} -split torus in G .

Ex. 1) $\text{rank}_{\mathbb{R}} SL(n, \mathbb{R}) = n-1$. Minor ignored technicality: what if we take a different embedding? One can show that this doesn't matter.

2) $\text{rank}_{\mathbb{R}} SL(n, \mathbb{C}) = \text{rank}_{\mathbb{R}} SL(n, \mathbb{H}) = n-1$

This is because the only matrices that remain diagonal under the embeddings $SL(n, \mathbb{C}) \hookrightarrow SL(2n, \mathbb{R})$ resp. $SL(n, \mathbb{H}) \hookrightarrow SL(4n, \mathbb{R})$ are real-valued diagonal matrices.

3) $\text{rank}_{\mathbb{R}} G = 0 \iff G$ is compact.

Prop. $\text{rank}_{\mathbb{R}} SO(m, n) = \min(m, n)$. In particular, $\text{rank}_{\mathbb{R}} SO(1, n) = 1$.

Pf. $(SO(1,1)^{\min(m,n)})^{\circ} < SO(m, n)$

$$\dim SO(1,1)^{\min(m,n)} = \min(m, n) \Rightarrow \text{rank}_{\mathbb{R}} SO(m, n) \geq \min(m, n)$$

NTS reversed inequality. (Coward's approach)

Let A be a maximal \mathbb{R} -split form in $SO(m, n)$.

$a \in A \setminus \{e\}$ since a is \mathbb{R} -diagonalisable and nontrivial

$$\Rightarrow \exists v \in \mathbb{R}^n \setminus \{0\}, \exists \lambda \in \mathbb{R} \setminus \{1\} \text{ s.t. } av = \lambda v$$

Take $a \approx e \rightarrow \lambda \neq -1$

$$\langle v, v \rangle_{\min} = \langle av, av \rangle_{\min} = \lambda^2 \langle v, v \rangle_{\min}$$

Since $\lambda^2 \neq 1$: $\langle v, v \rangle_{\min} = 0$

By the same argument: if $\dim(A) \geq k$ we get a totally isotropic subspace, (i.e. $V \subset \mathbb{R}^{m+n}$, $\langle v, v \rangle_{\min} = 0 \forall v \in V$) of dim k .

Since $\max \{ \dim V, \mid V \subset (\mathbb{R}^{m+n}, \langle \cdot, \cdot \rangle_{\min}) \text{ tot. isotr subsp.} \} = \min(m, n)$,

$\text{rank}_{\mathbb{R}} SO(m, n) \leq \min(m, n)$ as derived. □

Ch 16. Margulis supernrigidity theorem

NOT IN THE EXAM

Thm. Suppose $\text{rank}_{\mathbb{R}} G \geq 2$ and G is connected,

- $\Gamma < G$ irreducible
- H connected non-compact simple linear Lie group with trivial center
- $\varphi: \Gamma \rightarrow H$ homomorphism
- $\varphi(\Gamma) \subset H$ is Zariski dense

Then φ extends to a continuous homomorphism $\hat{\varphi}: G \rightarrow H$.

NB. This is not the most general version.

Applications (§16.2)

Thm. (Mostow rigidity) Suppose

- G_1, G_2 are connected with trivial center and no compact factors
- $PSL(2, \mathbb{R}) \not\cong G$
- $\Gamma_i < G_i$ irreducible lattices ($i=1, 2$)
- $\varphi: \Gamma_1 \rightarrow \Gamma_2$ a group iso

Then φ extends to a cont iso $\hat{\varphi}: G_1 \rightarrow G_2$.

Remark For rank 2 cases, e.g. for $SO(1, n)$ with $n \geq 3$, a different approach is needed.

Flat vector bundles

Def. $\varphi: \Gamma \rightarrow GL(n, \mathbb{R})$ a homomorphism. One obtains an action $\Gamma \curvearrowright G \times \mathbb{R}^n$:

$$(x, v) \gamma = (x\gamma, \varphi(\gamma^{-1})v)$$

Let $E_\varphi = G \times \mathbb{R}^n / \Gamma$. Then there is a map $\pi: E_\varphi \rightarrow G/\Gamma$ turning

E_φ into a vector bundle with fiber \mathbb{R}^n . This is called a flat vector bundle.

We formulate a variation of Margulis' theorem:

Thm. Assume

- $G = SL(n, \mathbb{R})$
- $\Gamma < G$ s.t. G/Γ is not compact
- $\varphi: \Gamma \rightarrow GL(n, \mathbb{R})$ any homomorphism.

Then there are

- $\hat{\varphi}: G \rightarrow GL(n, \mathbb{R})$ cont. homomorphism
- $\Gamma' < \Gamma$ finite index subgroup.

for which $\varphi(\gamma) = \hat{\varphi}(\gamma) \quad \forall \gamma \in \Gamma'$.

Prop. $G = SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z})$. If E_φ is any flat vector bundle over G/Γ then $\exists \Gamma' < \Gamma$ of finite index such that the lift $E_{\varphi'}$ of E_φ to the finite cover $G/\Gamma' \rightarrow G/\Gamma$ is trivial, i.e. isometric to $G/\Gamma' \times \mathbb{R}^n$.

PF OF PROP: Take Γ' as in the Theorem.

Define $T: G \times \mathbb{R}^n \rightarrow G \times \mathbb{R}^n$

$$(g, v) \mapsto (g, \hat{\varphi}(g)v)$$

$\forall \gamma \in \Gamma'$: $T((g, v)\gamma) = (g\gamma, \hat{\varphi}(g)v) \rightarrow T$ factors through the bundle isometry $E_{\varphi'} \cong G/\Gamma' \times \mathbb{R}^n$.

§ 16.3 Super-rigidity \rightarrow arithmeticity

Thm. (Margulis) $\text{rank}_{\mathbb{R}} G \geq 2$, $\Gamma < G$ irreducible lattice $\Rightarrow \Gamma$ is arithmetic.

Pf (SKETCH): Goal: show that we can assume $\Gamma < G_{\mathbb{Z}}$. Then we automatically get that Γ is commensurable to $G_{\mathbb{Z}}$, hence Γ is arithmetic.

10.07.2018

Step 1: Super-rigidity \Rightarrow the matrix entries of elements of Γ are algebraic.

Step 2: Use restriction of scalars to make them rational.

Step 3: Super-rigidity over $\mathbb{Q}_p \Rightarrow \forall p$ prime $\exists N_p$ universal upper bound, i.e. no matrix entry of any elt of Γ has a denominator divisible by a higher power of p . ("Denominators don't get too bad.")

Step 4: $\Rightarrow \exists \Gamma' < \Gamma$ finite index, contained in $G_{\mathbb{Z}}$.

Now we do these steps in detail.

Step 1: \uparrow Suppose $\exists \gamma \in \Gamma \exists i, j$ s.t. γ_{ij} is transcendental.

Galois thy $\Rightarrow \forall \alpha \in \mathbb{R} \setminus \overline{\mathbb{Q}} \exists \phi \in \text{Aut}(\mathbb{C}) : \phi(\gamma_{ij}) = \alpha$

We obtain an automorphism $\tilde{\phi} \in \text{Aut} SL(\ell, \mathbb{C})$ by applying ϕ to the entries.

\Rightarrow get a homomorphism $\rho := \tilde{\phi}|_{\Gamma} : \Gamma \rightarrow SL(\ell, \mathbb{C})$

Super-rigidity $\Rightarrow \exists \varphi : G \rightarrow SL(\ell, \mathbb{C})$ cont. homomorphism,

$\tilde{\phi} = \varphi$ on a finite index subgroup of Γ

\Rightarrow get uncountably many distinct homomorphisms $G \rightarrow SL(\ell, \mathbb{C})$ by varying α .

Lie thy + some extra argument $\Rightarrow \zeta$

Step 2: Γ finitely generated \Rightarrow the subfield of \mathbb{C} generated by the matrix entries is a finite extension of \mathbb{Q} .

Apply restriction of scalars.

Step 3: We need the following.

Thm. (Margulis super-rigidity / \mathbb{Q}_p) G connected, $\text{rank}_{\mathbb{R}} G \geq 2$,

- Γ irreducible,
- $\Gamma \xrightarrow{\varphi} SL(n, \mathbb{Q}_p)$ homomorphism

Then $\overline{\varphi(\Gamma)}$ is compact, i.e. $\exists N \in \mathbb{Z}$ s.t. \forall matrix entry of \forall elt of $\varphi(\Gamma)$ is in $p^N \mathbb{Z}_p$.

Since $\mathbb{Q} \subset \mathbb{Q}_p$, this completes Step 3.

Step 4. Let $D := \{\text{denominators of matrix entries of } \Gamma\} \subseteq \mathbb{N}$.

We know that for a given p , D only contains finitely many.

Wts D is bounded, i.e. only finitely many primes contribute to D .

Γ is finitely generated, let p_1, \dots, p_r be the primes appearing in its generators. Then every elt of D is of the form $p_1^{m_1} \dots p_r^{m_r}$ with $m_i \in \mathbb{N}$.

By Step 3, the m_i are bounded $\Rightarrow \exists M \in \mathbb{N} : \Gamma \subseteq \frac{1}{M} \text{Mat}_{l \times l}(\mathbb{Z})$

$\Rightarrow \exists$ fin. index $\Gamma' < \Gamma$ lying in $SL(l, \mathbb{Z})$. by Ex. 16.3.2.

The following has a similar proof:

Thm. (Commensurability criteria for arithmetic)

Assume G to be connected and Γ to be irreducible.

Then Γ is arithmetic $\iff \text{Conj}_G(\Gamma)$ is dense in G .

(topologically, not Zariski)

Lemma. $\varphi: \Gamma \rightarrow GL(n, \mathbb{R})$ extends to a continuous hom. $\tilde{\varphi}: G \rightarrow GL(n, \mathbb{R})$

iff $\exists V \in \text{Sect}(E_\varphi)$ G -invariant subspace s.t. $V \rightarrow V_{[\varphi]}$ is bijective where

$[\varphi] \in G/\Gamma$ is the img of φ in G/Γ .

Pf: " \Leftarrow " G -invariance of $V \Rightarrow$ get map $G \rightarrow GL(V)$

$V_{[\varphi]} = \mathbb{R}^n \Rightarrow V = V_{[\varphi]} = \mathbb{R}^n \Rightarrow$ get $G \rightarrow GL(n, \mathbb{R})$

To see that this extends φ note that $\text{Sect}(E_\varphi) \xleftrightarrow{1:1} (\text{right } \Gamma\text{-equivariant maps } G \rightarrow \mathbb{R}^n)$

If $\xi: G \rightarrow \mathbb{R}^n$ is right Γ -equivariant, i.e. $\xi(g\gamma) = \varphi(\gamma^{-1})\xi(g)$

then $\tilde{\xi}: G/\Gamma \rightarrow E_\varphi$ defined by $\tilde{\xi}(g\Gamma) := [(g, \xi(g))]$ is a well-def'd action.

" \Rightarrow " $v \in \mathbb{R}^n$, $g \in G$, define $\xi_v(g) := \tilde{\varphi}(g^{-1}) \cdot v$

This is a right Γ -equivariant map \rightarrow defines a section.

$V := \{ \xi_v \mid v \in \mathbb{R}^n \}$, $v \mapsto \xi_v$ is a linear G -equivariant map

$\Rightarrow V$ is a G -invariant subspace

Since $\xi_v(e) = \tilde{\varphi}(e)v = v$, we have the desired bijection.

Now we look at where higher rank comes into play.

Lemma. If $\text{rank}_{\mathbb{R}} G \geq 2$ then $\exists r \in \mathbb{N} \exists L_1, \dots, L_r < G$ closed subgroups s.t.

1) $G = L_r L_{r-1} \dots L_1$

2) both $H_i := L_i \cap A$, $H_i^\perp := C_A(L_i)$ are non-compact.

Pf: Lie groups. (Maybe next semester?)

Lemma. If we have

- $H < A$ closed non-compact subgroup
- $V \subset \text{Sect}(\tilde{E}_\varphi)$ H -invariant finite dimensional subspace

$\Rightarrow \langle C_G(H), V \rangle$ has finite dimension.

This lemma also goes into the pf of super-rigidity.

PROOF OF SUPERRIGIDITY:

Step 1: Find a nonzero section $\sigma: G/\Gamma \rightarrow \tilde{E}_\varphi$ which is A -invariant.

Set $V_0 := \mathbb{R}\sigma \subset \text{Sect} \tilde{E}_\varphi$

Since $H_1 = L_1 \cap A < A$, V_0 is H_1 -invariant

Using H_1^\perp and the prev lemma we get a new fin dim subspace of $\text{Sect}(\tilde{E}_\varphi)$, invariant under A, H_1, H_1^\perp .

Repeat \rightarrow get invariant subspace for G -action which is good enough.

Recall: sections of $\tilde{E}_\varphi \xrightarrow{1:1} (\text{right } \Gamma\text{-equivariant maps } G \rightarrow \mathbb{R}^n)$

Fact (§16.6) For some embedding $H \hookrightarrow \text{SL}(l, \mathbb{R})$:

- 1) H acts irreducibly on \mathbb{R}^l
- 2) $\exists \xi: G/A \rightarrow \mathbb{R}^n$ Γ -equivariant function, measurable, $\xi \neq 0$.

Something is amenable, use ergodic theory \rightsquigarrow